Stochastic Processes and their Applications 8 (1979) 269–276

CONVERGENCE OF BRANCHING TRANSPORT PROCESSES TO BRANCHING BROWNIAN MOTION

Luis G. GOROSTIZA* and Richard J. GRIEGO**
Centro de Investigación del I.P.N., México and the University of New Mexico, U.S.A.

Received 12 May 1978
Revised 2 January 1979

Two models are given of branching transport processes that converge to branching Brownian motion starting with one initial particle. The martingale problem method is used.

branching process	transport process
Brownian motion	weak convergence
martingale method	

1. Introduction

Bensoussan, Lions and Papanicolaou [1] have investigated asymptotics for branching transport processes. In their scaling the initial number of particles goes to infinity. Aside from the physical relevance of this scaling, it compensates for the collapse, due to the time scaling, of the branching process originated by each initial particle. The purpose of this note is to point out that it is also possible to scale in such a way that the branching transport process evolving from one initial particle converges to a branching diffusion. The idea is to have many more scatterings (direction changes only) than branchings. We present here two simple models, chosen to bring out the scattering and branching aspects. These models can be extended to several dimensions, space dependence, and more general motions between scatterings. We use the martingale problem method, adapting ideas of Papanicolaou, Stroock and Varadhan [5]. Thorough understanding of our paper requires reading the relevant parts of [5]. An area of related interest is branching Brownian motions with infinitely many initial particles (e.g. [2, 3]).

* Research supported in part by CONACYT grant PNCB 1140
** Research supported in part by NIH grant RR08139

2. Branching transport processes and limit theorems

We consider particles moving on the line $\mathbf{R}$, one initial particle starting from the origin at time 0 in direction $y(=1$ or $-1)$ and continuing with constant speed $v>0$. The particles will change direction and split according to the two models described below. Let $x(t)$ and $y(t)$ denote respectively the positions and directions of all particles present at time t. The state space for the branching transport process $(x(\cdot), y(\cdot))$ is $\bigcup_{n=1}^{\infty} E^{(n)}$, where $E^{(n)}$ is the symmetrized cartesian product of $E=\mathbf{R}\times\{1,-1\}$ with itself n times (we omit writing an extinction point).

Two time model. Let τ_1 and τ_2 be independent exponentially distributed random times with respective parameters λ_1 and λ_2, and let $\tau=\min(\tau_1,\tau_2)$. If $\tau=\tau_1$, at this time the initial particle going in direction y scatters in the following way: it continues in the same direction y with probability p or changes to direction $-y$ with probability $1-p$ $(p<1)$. If $\tau=\tau_2$, at this time the initial particle going in direction y branches in the following way: it splits into n particles with probability p_n, $n=0,2,3,\dots$, and the n $(n>1)$ new particles choose directions according to a distribution $\mu_y(\mathrm{d}y^1,\dots,\mathrm{d}y^n)$, depending on y, which, as will be seen, need not specify. The new particles evolve independnently in the same way.

We scale by replacing v and λ_1 by v/ε and λ_1/ε^2, $\varepsilon>0$ very small, and leave λ_2, i.e. we increase the speed and decrease the mean scattering time in the usual way but maintain the branching time. Let $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ denote the branching transport process so scaled.

Theorem 1. *The two time branching transport position process $x^\varepsilon(\cdot)$ converges weakly as $\varepsilon\to 0$ to branching Brownian motion with variance parameter $v^2/\lambda_1(1-p)$, particle lifetime distribution exponential with parameter λ_2, and particle production law p_n, $n=0,2,3,\dots$.*

Asymptotic criticality model. Let τ be an exponentially distributed random time with parameter λ. At time τ the initial particle going in direction y splits into n particles with probability p_n, $n=0,1,2,\dots$. The case $n=1$ represents a scattering but is considered here as a splitting. For $n=1$ the (new or same) particle continues in the same direction y with probability p or changes to direction $-y$ with probability $1-p$, and for $n>1$ the new particles choose directions according to a distribution $\mu_y(\mathrm{d}y^1,\dots,\mathrm{d}y^n)$ as before. The new particles evolve independently in the same way.

The scaling replaces v and λ by v/ε and λ/ε^2 as before, but in addition we change p_n to $p_n^{(\varepsilon)}$, with $(p_1^{(\varepsilon)})^{1/\varepsilon^2}\to a$, $0<a<1$, and $p_n^{(\varepsilon)}/\varepsilon^2\to a_n$, $n\neq 1$, as $\varepsilon\to 0$. The latter implies $p_1^{(\varepsilon)}\to 1$ and $p_n^{(\varepsilon)}\to 0$, $n\neq 1$, as $\varepsilon\to 0$, i.e. the particle production law is asymptotically critical. This is another way of having many more scatterings than branchings. Let $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ denote the scaled branching transport process.

Theorem 2. *The asymptotically critical branching transport position process $x^\varepsilon(\cdot)$ converges weakly as $\varepsilon \to 0$ to branching Brownian motion with variance parameter $v^2/\lambda(1-p)$, particle lifetime distribution exponential with parameter $-\lambda \log a$, and particle production law $\bar{p}_n = -a_n/\log a$, $n = 0, 2, 3, \ldots$.*

Remarks. (1) The two branching transport models can be obtained from each other and both theorems can be covered by a single one via an appropriate notation. However, we prefer to separate them in order to single out the two different scalings and different limits. In the asymptotic criticality model the particle lifetime distribution and particle production law in the branching transport process interact to yield the limiting lifetime distribution and production law; moreover, the asymptotic criticality of the production law refers to the branching transport process, and the production law of the limiting branching Brownian motion may be critical, subcritical or supercritical (this is because in the branching transport process a scattering, corresponding to $n = 1$, is taken as a special case of splitting, but in branching Brownian motion $n = 1$ is not considered).

(2) In both models the choice of directions after splitting according to μ_y is irrelevant in the limit. This is due to the decoupling in the limit of the position and velocity processes caused by the ergodicity of the direction process (see [5] for a discussion of decoupling).

3. Proofs

We use the martingale problem method [6] as in [5] (these works don't involve branching but their methods can be adapted), leaving to the reader the verification of precise definitions and standard technical details.

Two time model. One can show in a direct way that the infinitesimal generator $\mathscr{L}^\varepsilon$ of the Markov process $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ is

$$\mathscr{L}^\varepsilon = \frac{1}{\varepsilon}\mathscr{D} + \frac{1}{\varepsilon^2}\mathscr{J} + \mathscr{K},$$

where, for a real function f smooth with compact support on $\bigcup_n E^{(n)}$ properly topologized we have, restricted to $E^{(n)}$,

$$\mathscr{D}f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle) = v \sum_{k=1}^{n} y_k \frac{\partial f}{\partial x_k}(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle),$$

$$\mathscr{J}f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle) =$$

$$= \lambda_1 \sum_{k=1}^{n} [pf(\langle(x_1, y_1), \ldots, (x_k, y_k), \ldots, (x_n, y_n)\rangle)$$

$$+ (1-p)f(\langle(x_1, y_1), \ldots, (x_k, -y_k), \ldots, (x_n, y_n)\rangle)$$

$$-f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle)],$$

$$\mathscr{K}f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle) =$$

$$= \lambda_2 \sum_{k=1}^{n} \sum_{m\neq 1} p_m \int \ldots \int [f(\langle(x_1, y_1), \ldots, (x_k, y^1), \ldots, (x_k, y^m), \ldots$$

$$\ldots, (x_n, y_n)\rangle) - f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle)]\mu_{y_k}(\mathrm{d}y^1, \ldots, \mathrm{d}y^m).$$

$\mathscr{D}$ represent the linear motion, $\mathscr{I}$ the scattering and $\mathscr{K}$ the branching.

Now we apply the martingale problem method [5] (an adaptation of Theorem 2.1, p. 12, and Theorem 2.4, p. 44). We do some of the steps here as an illustration of this nice technique. Occasionally we restrict to $E^{(1)}$ to simplify notation.

Let $P(t, y, A; x)$ denote the probability transition function of the Markov process generated by $\mathscr{I}$ with x considered as a parameter, i.e.

$$P(t, y, \{\pm y\}; x) = \tfrac{1}{2}(1 \pm \mathrm{e}^{-2\lambda_1(1-p)t}).$$

This is an ergodic process with invariant measure

$$\bar{P}(\{y\}; x) = \begin{cases} \tfrac{1}{2} & \text{if } y = 1, \\ \tfrac{1}{2} & \text{if } y = -1, \end{cases}$$

the centering condition $\int y\bar{P}(\mathrm{d}y; x) = 0$ holds, and the recurrent potential kernel is ([5], p. 43)

$$\psi(y, \{\pm y\}; x) = \int_0^\infty [P(t, y, \{\pm y\}; x) - \bar{P}(\{\pm y\}; x)]\,\mathrm{d}t$$

$$= \pm \frac{1}{4\lambda_1(1-p)}.$$

Let f be a real smooth function with compact support on $\bigcup_{n=1}^\infty \mathbf{R}^{(n)}$, i.e., depending on positions only, and define f^ε on $\bigcup_{n=1}^\infty E^{(n)}$ by [5, p. 20]

$$f^\varepsilon(x, y) = f(x) + \varepsilon\psi_f^{(1)}(x, y) + \varepsilon^2\psi_f^{(2)}(x, y),$$

where

$$\psi_f^{(1)}(x, y) = \int \psi(y, \mathrm{d}z; x)vz\frac{\partial f}{\partial x}(x) = \frac{vy}{2\lambda_1(1-p)}\frac{\partial f}{\partial x}(x),$$

and

$$\psi_f^{(2)}(x, y) = \int \psi(y, \mathrm{d}z; x)A_f(x, z),$$

with, letting $g(x, y) = f(x)$,

$$A_f(x, y) = \mathscr{D}\psi_f^{(1)}(x, y) + \mathscr{K}g(x, y)$$

$$= \frac{v^2y^2}{2\lambda_1(1-p)}\frac{\partial^2 f}{\partial x^2}(x) + \lambda_2 \sum_{m\neq 1} p_m \int \cdots \int [g(\langle(x, y^1), \ldots, (x, y^m)\rangle)$$

$$-g(\langle(x, y)\rangle)]\mu_y(\mathrm{d}y^1, \ldots, \mathrm{d}y^m)$$

$$= \frac{v^2}{2\lambda_1(1-p)}\frac{\partial^2 f}{\partial x^2}(x) + \lambda_2 \sum_{m\neq 1} p_m[f(\langle\underbrace{x, \ldots, x}_{m \text{ times}}\rangle) - f(\langle x\rangle)],$$

which we denote $A_f(x)$ because it does not depend on y. (Observe that μ_y disappears because g does not depend on the y^i).

Define ([5], p. 43)

$$\bar{\mathscr{L}}f(x)=\int \bar{P}(\mathrm{d}z;x)A_f(x,z),$$

hence $\bar{\mathscr{L}}f(x)=A_f(x)$. In general, on $E^{(n)}$,

$$\bar{\mathscr{L}}f(\langle x_1,\dots,x_n\rangle)=\frac{v^2}{2\lambda_1(1-p)}\sum_{k=1}^{n}\frac{\partial^2 f}{\partial x_k^2}(\langle x_1,\dots,x_n\rangle)$$

$$+\lambda_2\sum_{k=1}^{n}\sum_{m\neq 1}p_m[f(\langle x_1,\dots,\underbrace{x_k,\dots,x_k}_{m\text{ times}},\dots,x_n\rangle)-f(\langle x_1,\dots,x_n\rangle)].$$

We are following the procedure indicated in the remark in [5, p. 19]:

$$\begin{aligned}\mathscr{L}^\varepsilon f^\varepsilon &=\left(\frac{1}{\varepsilon}\mathscr{D}+\frac{1}{\varepsilon^2}\mathscr{I}+\mathscr{K}\right)(f+\varepsilon\psi_f^{(1)}+\varepsilon^2\psi_f^{(2)})\\ &=\frac{1}{\varepsilon^2}\mathscr{I}f+\frac{1}{\varepsilon}(\mathscr{D}f+\mathscr{I}\psi_f^{(1)})+(\mathscr{D}\psi_f^{(1)}+\mathscr{I}\psi_f^{(2)}+\mathscr{K}f-\bar{\mathscr{L}}f)\\ &\quad+\bar{\mathscr{L}}f+\varepsilon(\mathscr{D}\psi_f^{(2)}+\mathscr{K}\psi_f^{(1)})+\varepsilon^2\mathscr{K}\psi_f^{(2)},\end{aligned}$$

where $\mathscr{I}f=0$ because $\mathscr{I}$ operates on y only, and $\psi_f^{(1)}$, $\psi_f^{(2)}$ are the solution of

$$\mathscr{D}f+\mathscr{I}\psi_f^{(1)}=0,$$
$$\mathscr{D}\psi_f^{(1)}+\mathscr{I}\psi_f^{(2)}+\mathscr{K}f-\bar{\mathscr{L}}f=0.$$

Hence

$$\mathscr{L}^\varepsilon f^\varepsilon=\bar{\mathscr{L}}f+\mathrm{O}(\varepsilon).$$

Weak sequential compactness of $\{x^\varepsilon(\cdot)\}$ can be proved according to the following argument: Weak sequential compactness of $\{x^\varepsilon(\cdot)\}$ is destroyed by large oscillations in small time intervals. Now, oscillations of $x^\varepsilon(\cdot)$ are due to oscillations of the process restricted to $\mathbf{R}^{(n)}$, n fixed (i.e., no branching) or to branchings (changes in n). For $\{x^\varepsilon(\cdot)\}$ restricted to $\mathbf{R}^{(n)}$ the method of [5, p. 17] applies and gives compactness, and the branchings don't produce large oscillations in small time intervals because the times between consecutive branchings don't go to 0.

To identify the limit we have that, because $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ is a Markov process with infinitesimal generator $\mathscr{L}^\varepsilon$,

$$M_{f^\varepsilon}(t)=f^\varepsilon(x^\varepsilon(t),y^\varepsilon(t))-f^\varepsilon(x,y)-\int_0^t\mathscr{L}^\varepsilon f^\varepsilon(x^\varepsilon(r),y^\varepsilon(r))\,\mathrm{d}r$$

is a martingale starting at $f^\varepsilon(x, y)$. Substituting $f^\varepsilon = f + \varepsilon\psi_f^{(1)} + \varepsilon^2\psi_f^{(2)}$ and $\mathcal{L}^\varepsilon f^\varepsilon = \bar{\mathcal{L}}f + O(\varepsilon)$ we obtain, for $s < t$,

$$f(x^\varepsilon(t)) - f(x^\varepsilon(s)) - \int_s^t \bar{\mathcal{L}}f(x^\varepsilon(r))\,dr = M_{f^\varepsilon}(t) - M_{f^\varepsilon}(s) + O(\varepsilon).$$

It follows as in [5, p. 18] that for any weak limit $x(\cdot)$ of $\{x^\varepsilon(\cdot)\}$ as $\varepsilon \to 0$,

$$f(x(t)) - f(x) - \int_0^t \bar{\mathcal{L}}f(x(r))\,dr$$

is a martingale starting at $f(x)$, i.e. $x(\cdot)$ is a diffusion with infinitesimal generator $\bar{\mathcal{L}}$, i.e. $x(\cdot)$ is the branching Brownian motion described in the theorem (see [3] for the uniqueness of the solution of the martingale problem corresponding to a generator of the type of $\bar{\mathcal{L}}$).

Remarks. (1) for f of the form $f(\langle x_1, \ldots, x_n\rangle) = f(x_1)\ldots f(x_n)$, our linear generator $\bar{\mathcal{L}}$ reduces to the nonlinear type considered by Ikeda, Nagasawa, Watanabe [4].

(2) For scattering in several dimensions, in general one does not have explicitly the transition function for $\mathcal{J}$ and hence the recurrent potential kernel; therefore our present method of proof, which consists essentially in following the steps of [5], cannot be extended directly.

Asymptotic criticality model. The infinitesimal generator of $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ is

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{D} + \frac{1}{\varepsilon^2}\mathcal{J}^\varepsilon + \mathcal{K}^\varepsilon,$$

where

$$\mathcal{D}f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle) = v \sum_{k=1}^n y_k \frac{\partial f}{\partial x_k}(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle),$$

$$\begin{aligned}&\mathcal{J}^\varepsilon f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle)\\ &\quad= p_1^{(\varepsilon)}\lambda \sum_{k=1}^n [pf(\langle(x_1, y_1), \ldots, (x_k, y_k), \ldots, (x_n, y_n)\rangle)\\ &\qquad\qquad + (1-p)f(\langle(x_1, y_1), \ldots, (x_k, -y_k), \ldots, (x_n, y_n)\rangle)\\ &\qquad\qquad - f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle)],\end{aligned}$$

$$\begin{aligned}&\mathcal{K}^\varepsilon f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle)\\ &\quad= \lambda \sum_{k=1}^n \sum_{m\neq 1} \frac{p_m^{(\varepsilon)}}{\varepsilon^2} \int\cdots\int [f(\langle(x_1, y_1), \ldots, (x_k, y^1), \ldots, (x_k, y^m), \ldots\\ &\qquad\qquad \ldots, (x_n, y_n)\rangle) - f(\langle(x_1, y_1), \ldots, (x_n, y_n)\rangle)]\mu_{y_k}(dy^1, \ldots, dy^m).\end{aligned}$$

(Observe that the operators $\mathscr{J}^{\varepsilon}$ and $\mathscr{K}^{\varepsilon}$ now have an ε dependence, but this causes no problem in taking the limit.)

The same procedure used above shows that $x^{\varepsilon}(\cdot)$ converges weakly to branching Brownian motion with infinitesimal generator

$$\bar{\mathscr{L}}f(\langle x_1,\ldots,x_n\rangle)=\frac{v^2}{2\lambda(1-p)}\sum_{k=1}^{n}\frac{\partial^2 f}{\partial x_k^2}(\langle x_1,\ldots,x_n\rangle)$$
$$+\lambda\sum_{k=1}^{n}\sum_{m\neq 1}a_m[f(\langle x_1,\ldots,x_k,\ldots,x_k,\ldots,x_n\rangle)$$
$$-f(\langle x_1,\ldots,x_n\rangle)].$$

In order to verify the statement concerning the limiting particle lifetime distribution, let $N(t)$ denote the renewal function associated with τ, i.e. $N(t)=\max\{k:\sum_{i=1}^{k}\tau_i\leq t\}$, where $\tau_1,\tau_2,\ldots,$ are independent and distributed as τ, and let τ^{ε} denote the time until extinction ($n=0$) or a real split ($n\geq 2$) of the initial particle in the $(x^{\varepsilon}(\cdot),y^{\varepsilon}(\cdot))$ process. It is easy to show by conditioning on the number of scatterings that

$$P[\tau^{\varepsilon}>t]=E(p_1^{(\varepsilon)})^{N(t/\varepsilon^2)}=E[(p_1^{(\varepsilon)})^{1/\varepsilon^2}]^{\varepsilon^2 N(t/\varepsilon^2)},$$

hence, using the renewal theorem,

$$\lim_{\varepsilon\to 0}P[\tau^{\varepsilon}>t]=a^{\lambda t}=\mathrm{e}^{-(-\lambda\log a)t},$$

i.e. τ^{ε} converges weakly to the exponential distribution with parameter $-\lambda\log a$.

For the limiting particle production law we note first that

$$(p_1^{(\varepsilon)})^{1/\varepsilon^2}=\left(1-\frac{\sum_{m\neq 1}p_m^{(\varepsilon)}/\varepsilon^2}{1/\varepsilon^2}\right)^{1/\varepsilon^2},$$

then, taking limit as $\varepsilon\to 0$ we get $\sum_{m\neq 1}a_m=-\log a$. So, denoting $\bar{p}_m$ the limiting probability of splitting into m particles, $m\neq 1$, we have

$$\begin{aligned}\bar{p}_m&=\lim_{\varepsilon\to 0}P[\text{split into } m \text{ particles}\,|\,m\neq 1]\\&=\lim_{\varepsilon\to 0}\frac{p_m^{(\varepsilon)}}{\sum_{m\neq 1}p_m^{(\varepsilon)}}\\&=-\frac{a_m}{\log a}.\end{aligned}$$

Finally, in the above expression for $\bar{\mathscr{L}}$ multiply λ by $-\log a$ and divide a_m by $-\log a$ in the branching part of the operator.

Acknowledgment

We wish to thank G. Papanicolaou and D. Stroock, with whom we have had fruitful conversations on the present problem. The referee made comments that improved the paper.

References

[1] A. Bensoussan, J.L. Lions and G.C. Papanicolaou, Asymptotics for branching transport processes, in: Proc. IRIA Congress (1977) to appear.

[2] D. Dawson and G. Ivanoff, Branching diffusions and random measures, in: A. Joffe and P. Ney, Eds., Branching Processes, Advances in Probability and Related Topics, Vol. 5 (Dekker, New York, 1978).

[3] R.A. Holley and D.W. Stroock, Generalized Ornstein–Uhlenbeck Processes and Infinite Particle Branching Brownian motions, to appear.

[4] N. Ikeda, M. Nagasawa and S. Watanabe, Branching Markov processes, J. Math. Kyoto Univ. 8 (1968) 233–278, 356–410, and 9 (1969) 95–160.

[5] G.C. Papanicolaou, D.W. Stroock and S.R.S. Varadhan, Martingale approach to some limit theorems, in: 1976 Duke Turbulence Conference (Duke Univ. Math. Dept., 1977).

[6] D.W. Stroock and S.R.S. Varadhan, Diffusion processes with continuous coefficients I, II, Comm. Pure Appl. Math. 22 (1969) 345–400 and 479–530.